An Introduction to Non-Euclidean Geometry
Nate Black

Clemson University
Math Science Graduate Student Seminar
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Euclid’s Common Notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.
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Euclid’s Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
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Euclid’s Propositions

- The 48 propositions are accompanied by a proof using the common notions, postulates, and previous propositions.

- The 29th proposition states:

  A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

- The 29th proposition is the first to make use of the 5th postulate.
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The 5th Postulate

- The Parallel Postulate
  - Playfair’s Axiom
    Through a point not on a given line there passes not more than one parallel to the line.
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Proving the 5th Postulate

- Posidonius (1st Century B.C.)
- Ptolemy (2nd Century A.D.)
- Proclus (5th Century A.D.)
- Many others...
- Saccheri (1667-1733)
  - Proof by Contradiction
  - Saccheri Quadrilateral
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The Saccheri Quadrilateral

- $AD = BC$
- $AD \perp AB$
- $BC \perp AB$
# Euclidean Geometry

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- We can model this with a negative curvature of space.
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Euclidean Model
Modelling with a saddle
Saddle Model Top View
Replacement of the 5th Postulate
The summit angles of a Saccheri quadrilateral are acute.

Thm: The summit angles of a Saccheri quadrilateral are equal.
Proof: Triangles $ABC$ and $BAD$ are congruent by SAS. Thus, $AC = BD$ and $\angle ADC = \angle BCD$. 
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\[ \begin{array}{c}
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![Diagram](image-url)
A parallel with a common perpendicular

Thm: The midline of a Saccheri quadrilateral is perpendicular to both the base and the summit.

Proof: Triangles $AED$ and $BEC$ are congruent by SAS. This implies that $ED = EC$ and triangles $DEF$ and $CEF$ are congruent by SSS. Thus, $\angle DFE = \angle CFE$, similarly one can show $\angle AEF = \angle BEF$. 
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![Diagram of a Saccheri quadrilateral with A parallel and a common perpendicular]
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Parallels with a common perpendicular

**Thm:** There are an infinite number of parallels with a common perpendicular passing through any point not on the line.

**Proof:** Take a point $L_1$ on $h$ to the right of $F$, let $M_1 = \text{Proj}_g(L_1)$. Take $P_1$ on $M_1L_1$ such that $\overline{EF} = \overline{M_1P_1}$. Then $EM_1P_1F$ is a Saccheri quadrilateral with summit lying on line $k_1$ and the midline is perpendicular to $g$ and $k_1$. Thus, $k_1$ is another parallel with a common perpendicular that passes through $F$. 
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**Parallels without a common perpendicular**

- **Thm:** For a given line, \( g \), and a point, \( F \), not on that line there exist 2 lines which are parallel to \( g \) and pass through \( F \) without a common perpendicular.

- **Proof:** Consider the set of all lines subdividing the right angle formed by the intersection of \( EF \) and \( h \). Then any of these lines either intersects \( g \) or is parallel to \( g \). Let \( I \) be the set of lines that intersect \( g \) and \( P \) be the set of lines that are parallel to \( g \). Consider the line, \( k \), that forms the boundary between these two sets. (ie. every line in \( I \) precedes \( k \), and \( k \) precedes every line in \( P \)) Suppose \( k \in I \), then \( k \) intersects \( g \) at some point, \( A \). If we take a point, \( B \), to the right of \( A \), then \( k \) precedes the line passing through \( F \) and \( B \). This cannot be, since every line in \( I \) precedes \( k \). Thus, \( k \in P \). Now \( k \) cannot be parallel with a common perpendicular since none of these lines make a smallest angle with \( EF \).
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Right Triangles

- **Thm**: Right triangles have angle sums < 180°.
- **Pf**: Consider a right triangle $ABC$, with a right angle at $A$. Let $h$ be the line that passes through $C$ so as to make $\angle 1 = \angle 2$. Then $g$ and $h$ are parallel with a common perpendicular that bisects $BC$. Clearly, $\angle 1 + \angle 3 = \angle 2 + \angle 3 < 90°$ since the angle that $AC$ makes with $h$ is acute.
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Triangles

- Any triangle can be decomposed into two right triangles both of which have angle sum less than 180°.
- Therefore, any triangle has angle sum less than 180°.
- The difference between the angle measure of a triangle and 180° is called the defect of the triangle. Smaller triangles have smaller defects and larger triangles have larger defects.
- The area of a triangle is proportional to its defect. (ie. \( A = kD \), where \( k \) is some positive constant and \( D \) is the defect of the triangle)
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# Hyperbolic vs. Euclidean Geometry

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Double Elliptic Geometry

- All lines are great circles, and thus all lines have the same length. We will assume the sphere has a radius of $k$ so the length of any line is $2\pi k$.
- Consequently, there is a maximum distance that any two points can be apart. Namely, half of the length of a line or $\pi k$.
- Any two lines meet in two points.
- Through each pair of nonpolar points, there passes exactly one line.
- Through each pair of polar points, there pass infinitely many lines.
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Double Elliptic Geometry

- All lines are great circles, and thus all lines have the same length. We will assume the sphere has a radius of $k$ so the length of any line is $2\pi k$.
- Consequently, there is a maximum distance that any two points can be apart. Namely, half of the length of a line or $\pi k$.
- Any two lines meet in two points.
- Through each pair of nonpolar points, there passes exactly one line.
- Through each pair of polar points, there pass infinitely many lines.
Spherical Lines
Thm: In a right triangle, the other angles are acute, right, or obtuse as the side opposite the angle is less than, equal to, or greater than $\frac{\pi k}{2}$. The converse is also true.

Proof: By diagram
Angle sum of Triangles

- Right triangles with another right angle or an obtuse angle clearly have an angle sum greater than 180°.
- Right triangles with only one acute angle have a third angle that is either right or obtuse, so these triangles have an angle sum greater than 180°.
- It can be shown that a right triangle with 2 acute angles has an angle sum greater than 180°.
- Since any triangle can be decomposed into 2 right triangles, we conclude that all triangles have angle sum greater than 180°.
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Modelling with a modified hemisphere

- The model for Single Elliptic Geometry is the modified hemisphere.
Single Elliptic Geometry

- All lines are great circles and have the same length. Since we are working with half a sphere this will be $\pi k$.
- Consequently, there is a maximum distance that any two points can be apart, namely $\frac{\pi k}{2}$.
- Any two lines meet in one point.
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Consequently, there is a maximum distance that any two points can be apart, namely $\frac{\pi k}{2}$.

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Any two lines meet in one point.
Triangles

- We can get some odd looking triangles though.
## Non-Euclidean vs. Euclidean Geometry

<table>
<thead>
<tr>
<th></th>
<th>Euclidean</th>
<th>Hyperbolic</th>
<th>Elliptic</th>
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<tbody>
<tr>
<td>Number of Parallels</td>
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<tr>
<td>Saccheri Angle Sum</td>
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<td>$&lt; \pi$</td>
<td>$&gt; \pi$</td>
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<tr>
<td>Curvature of space</td>
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<tr>
<td>Triangle Angle Sum</td>
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<td>$&lt; \pi$</td>
<td>$&gt; \pi$</td>
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<tr>
<td>Similar Triangles</td>
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<td>all congruent</td>
<td>all congruent</td>
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<tr>
<td>Extent of lines</td>
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<td>infinite</td>
<td>finite</td>
</tr>
</tbody>
</table>
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