An Introduction to Non-Euclidean Geometry

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Math Science Graduate Student Seminar

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Euclid's Elements

Euclid's Common Notions

- 1. Things which equal the same thing also equal one another.
- 2. If equals are added to equals, then the wholes are equal.
- 3. If equals are subtracted from equals, then the remainders are equal.
- 4. Things which coincide with one another equal one another.
- 5. The whole is greater than the part.

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Euclid's Postulates

- 1. To draw a straight line from any point to any point.
- 2. To produce a finite straight line continuously in a straight line.
- 3. To describe a circle with any center and radius.
- 4. That all right angles equal one another.
- 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

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Euclid's Elements

Euclid's Propositions

- The 48 propositions are accompanied by a proof using the common notions, postulates, and previous propositions.
- The 29th proposition states:

  > *A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.*

- The 29th proposition is the first to make use of the 5th postulate.

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- The 29th proposition is the first to make use of the 5th postulate.

The 5th Postulate

- The Parallel Postulate
- Playfair's Axiom
  Through a point not on a given line there passes not more than one parallel to the line.

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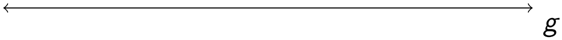

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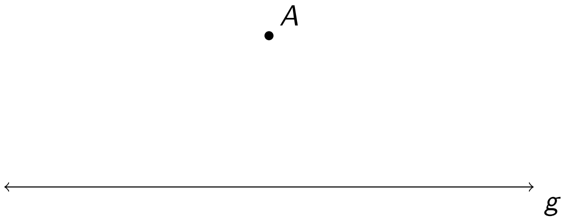

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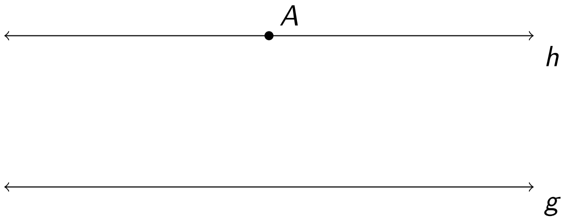

Proving the 5th Postulate

- Posidonius (1st Century B.C.)
- Ptolemy (2nd Century A.D.)
- Proclus (5th Century A.D.)
- Many others...
- Saccheri (1667-1733)
  - Proof by Contradiction
  - Saccheri Quadrilateral

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The Saccheri Quadrilateral

- $\overline{AD} = \overline{BC}$
- $AD \perp AB$
- $BC \perp AB$

Euclidean Geometry

	Euclidean
Number of Parallels	1
Saccheri Angle Sum	$= \pi$
Curvature of space	none
Triangle Angle Sum	$= \pi$
Similar Triangles	some congruent
Extent of lines	infinite

Modifying the 5th Postulate

- Suppose we change the 5th Postulate to read as follows: Through a point not on a given line there passes more than one parallel to the line.
- We can model this with a negative curvature of space.

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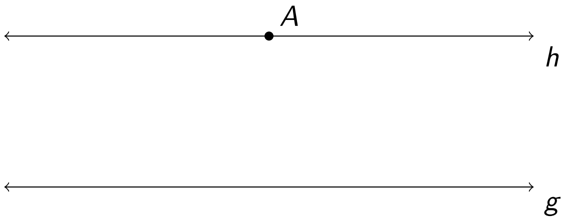

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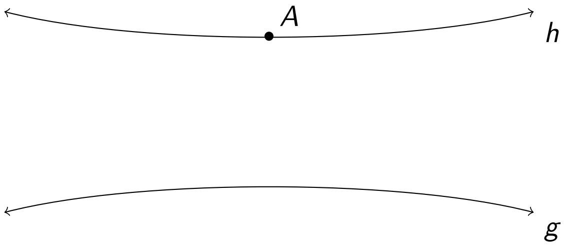

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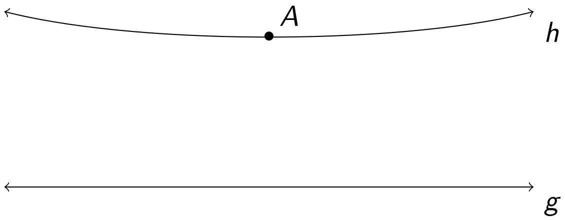
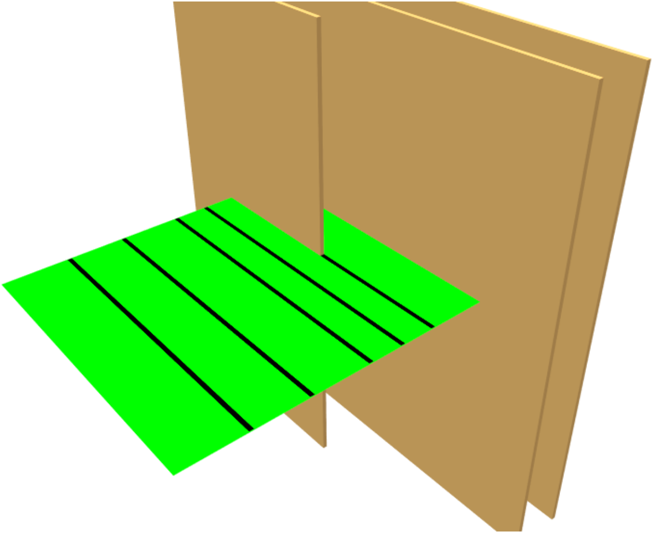
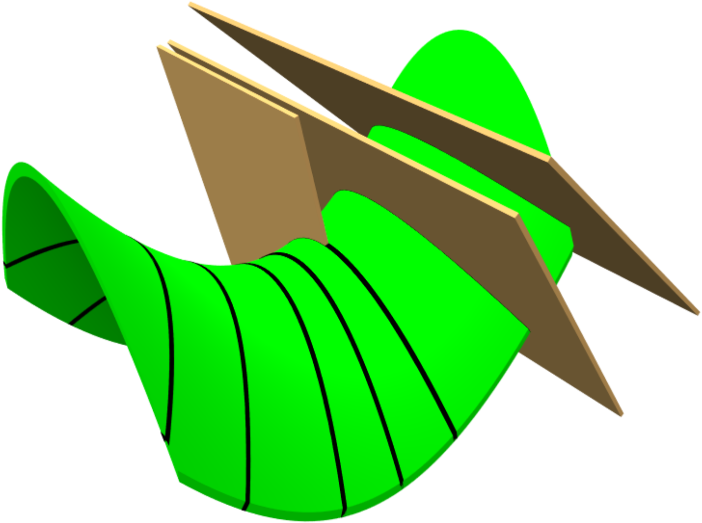


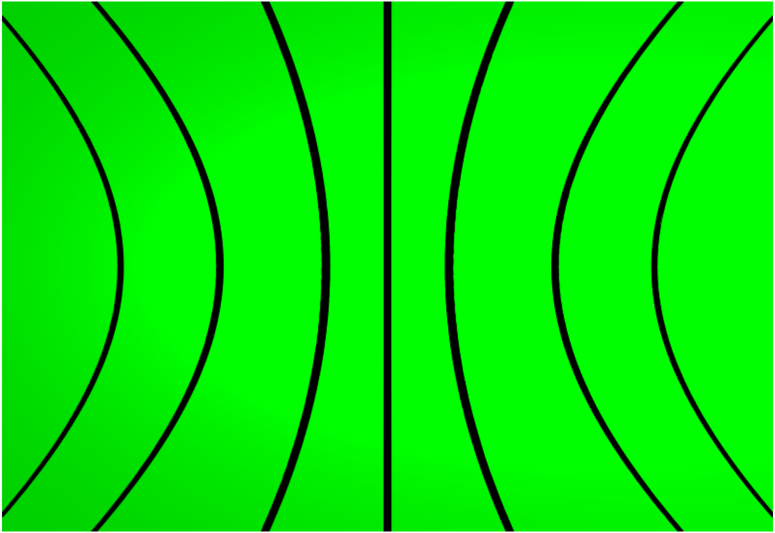
Euclidean Model



Modelling with a saddle



Saddle Model Top View



Hyperbolic Geometry

- Replacement of the 5th Postulate
  The summit angles of a Saccheri quadrilateral are acute.
- Thm: The summit angles of a Saccheri quadrilateral are equal.
  Proof: Triangles ABC and BAD are congruent by SAS. Thus, $\overline{AC} = \overline{BD}$ and $\angle ADC = \angle BCD$.

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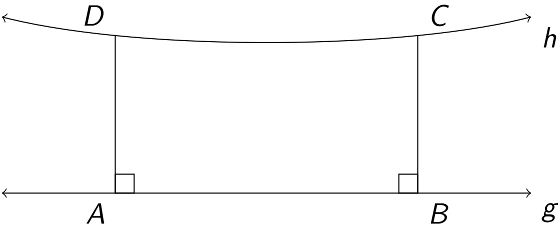

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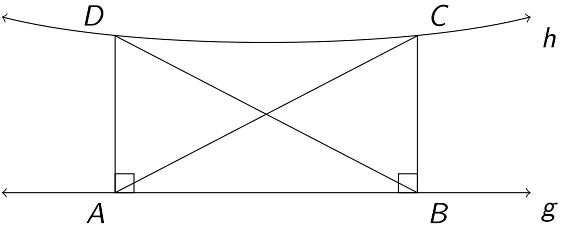

A parallel with a common perpendicular

- Thm: The midline of a Saccheri quadrilateral is perpendicular to both the base and the summit.
- Proof: Triangles AED and BEC are congruent by SAS. This implies that $\overline{ED} = \overline{EC}$ and triangles DEF and CEF are congruent by SSS. Thus, $\angle DFE = \angle CFE$, similarly one can show $\angle AEF = \angle BEF$.

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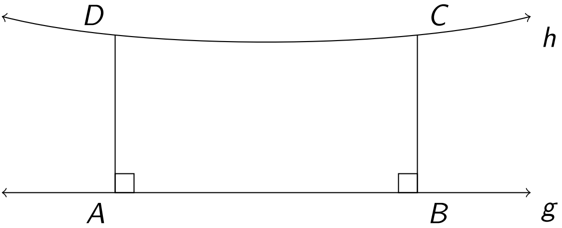

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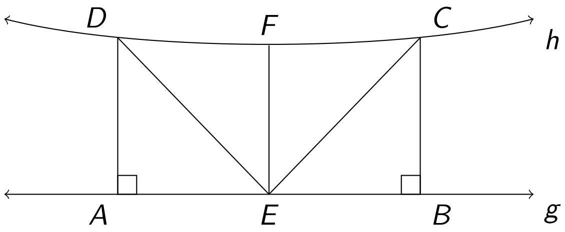

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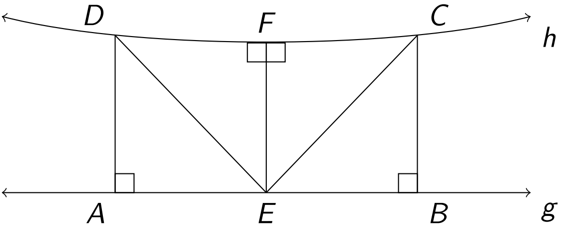

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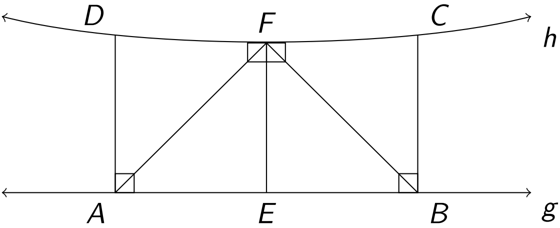

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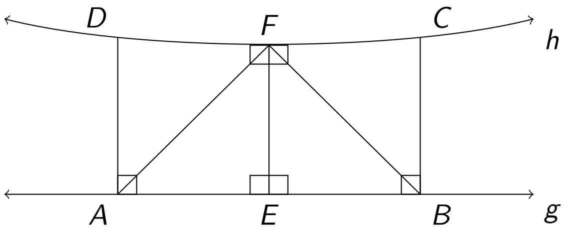

Parallels with a common perpendicular

- Thm: There are an infinite number of parallels with a common perpendicular passing through any point not on the line.
- Proof: Take a point L_1 on h to the right of F, let $M_1 = Proj_g(L_1)$. Take P_1 on M_1L_1 such that $\overline{EF} = \overline{M_1P_1}$. Then EM_1P_1F is a Saccheri quadrilateral with summit lying on line k_1 and the midline is perpendicular to g and k_1. Thus, k_1 is another parallel with a common perpendicular that passes through F.

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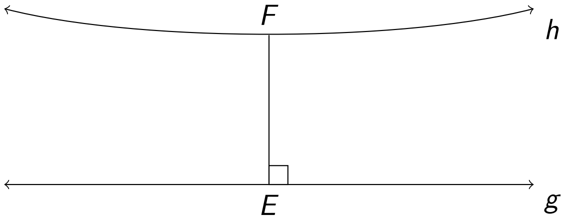

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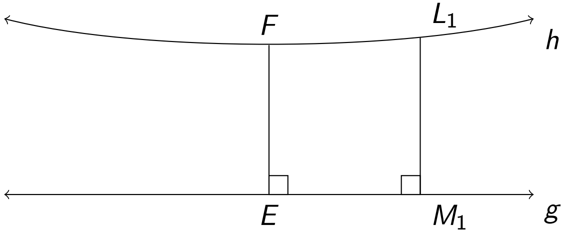

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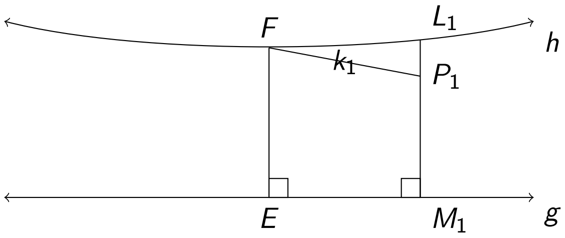

Parallels without a common perpendicular

- Thm: For a given line, g, and a point, F, not on that line there exist 2 lines which are parallel to g and pass through F without a common perpendicular.
- Proof: Consider the set of all lines subdividing the right angle formed by the intersection of EF and h. Then any of these lines either intersects g or is parallel to g. Let I be the set of lines that intersect g and P be the set of lines that are parallel to g. Consider the line, k, that forms the boundary between these two sets. (ie. every line in I precedes k, and k precedes every line in P) Suppose $k \in I$, then k intersects g at some point, A. If we take a point, B, to the right of A, then k precedes the line passing through F and B. This cannot be, since every line in I precedes k. Thus, $k \in P$. Now k cannot be parallel with a common perpendicular since none of these lines make a smallest angle with EF.

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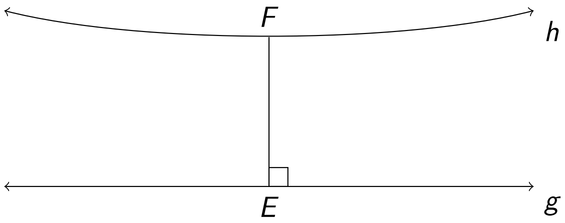

- These lines are called boundary parallels, and the angle α is called the angle of parallelism for F and g.

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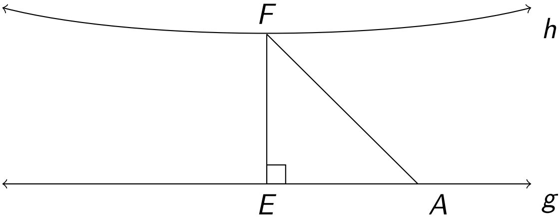

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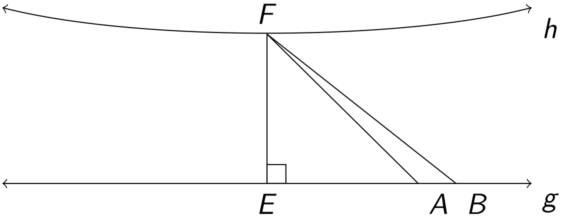

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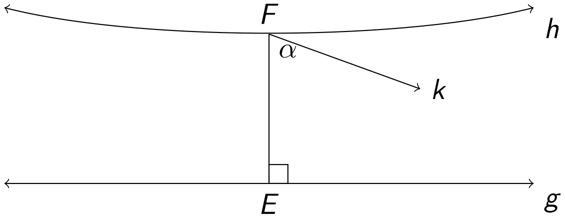

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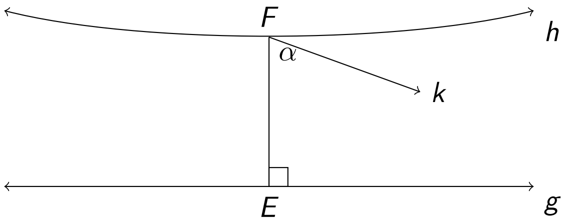

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Right Triangles

- Thm: Right triangles have angle sums $< 180°$.
- Pf: Consider a right triangle ABC, with a right angle at A. Let h be the line that passes through C so as to make $\angle 1 = \angle 2$. Then g and h are parallel with a common perpendicular that bisects BC. Clearly, $\angle 1 + \angle 3 = \angle 2 + \angle 3 < 90°$ since the angle that AC makes with h is acute.

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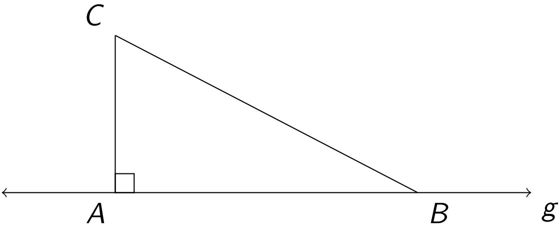

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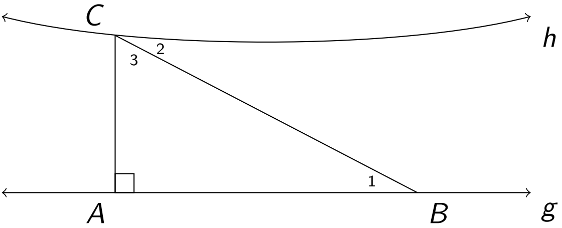

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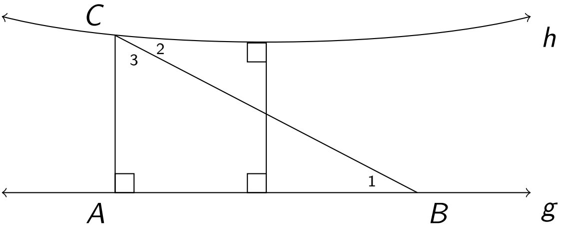

Triangles

- Any triangle can be decomposed into two right triangles both of which have angle sum less than 180°.
- Therefore, any triangle has angle sum less than 180°.
- The difference between the angle measure of a triangle and 180° is called the defect of the triangle. Smaller triangles have smaller defects and larger triangles have larger defects.
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- A trilateral is a three sided figure consisting of two boundary parallels and a transversal that cuts them both.
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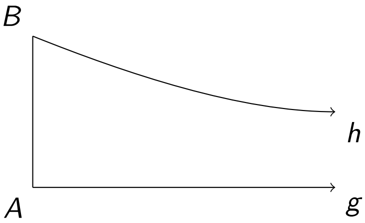

Hyperbolic vs. Euclidean Geometry

	Euclidean	Hyperbolic
Number of Parallels	1	∞
Saccheri Angle Sum	$= \pi$	$< \pi$
Curvature of space	none	negative
Triangle Angle Sum	$= \pi$	$< \pi$
Similar Triangles	some congruent	all congruent
Extent of lines	infinite	infinite

Modifying the 5th Postulate

- Suppose we change the 5th Postulate to read as follows:
  All lines intersect and thus, there are no lines which are parallel.
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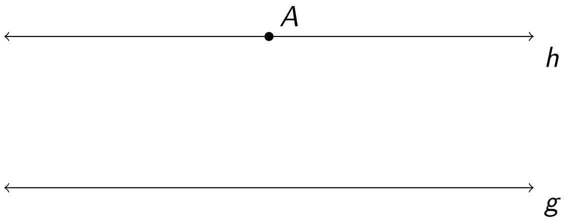

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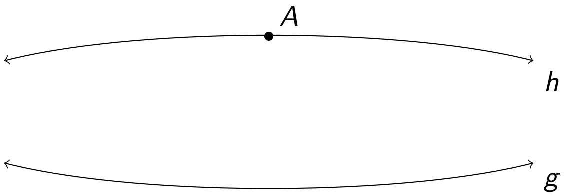

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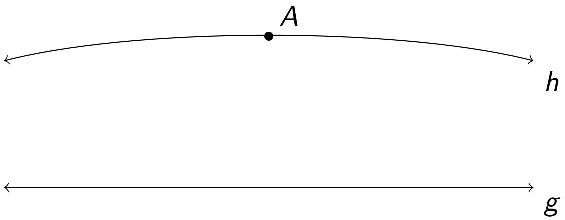

Elliptic Geometry

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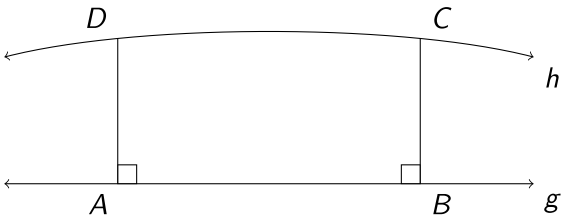
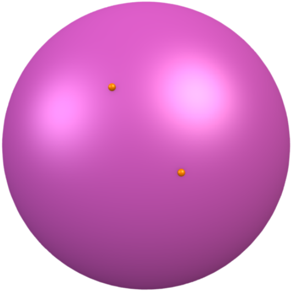


Modeling with a sphere

- The most familiar model for Elliptic Geometry is the sphere, technically this is the model for Double Elliptic Geometry.



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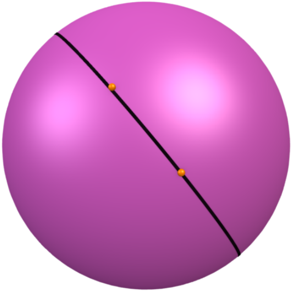

Double Elliptic Geometry

- All lines are great circles, and thus all lines have the same length. We will assume the sphere has a radius of k so the length of any line is $2\pi k$.
- Consequently, there is a maximum distance that any two points can be apart. Namely, half of the length of a line or πk.
- Any two lines meet in two points.
- Through each pair of nonpolar points, there passes exactly one line.
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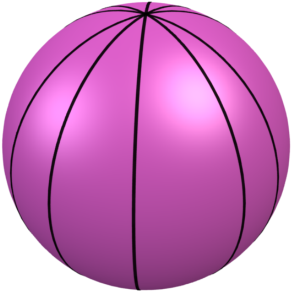
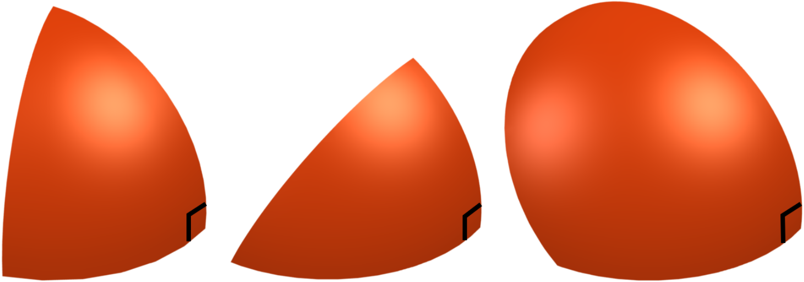


Right Triangles

- Thm: In a right triangle, the other angles are acute, right, or obtuse as the side opposite the angle is less than, equal to, or greater than $\frac{\pi k}{2}$. The converse is also true.
- Proof: By diagram



Angle sum of Triangles

- Right triangles with another right angle or an obtuse angle clearly have an angle sum greater than 180°.
- Right triangles with only one acute angle have a third angle that is either right or obtuse, so these triangles have an angle sum greater than 180°.
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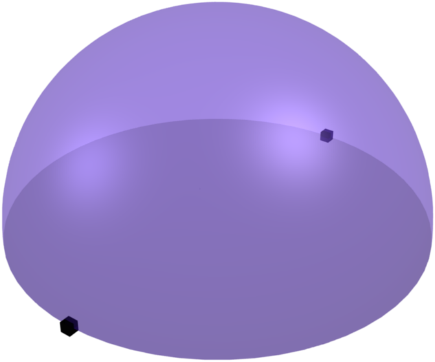
Modelling with a modified hemisphere

- The model for Single Elliptic Geometry is the modified hemisphere.



Single Elliptic Geometry

- All lines are great circles and have the same length. Since we are working with half a sphere this will be πk.
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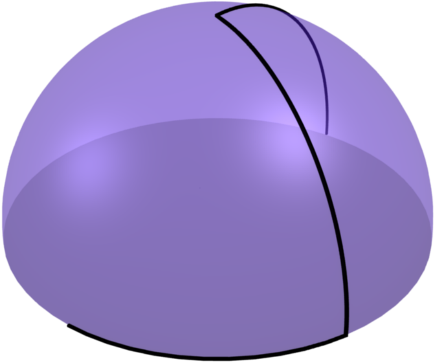
Triangles

- We can get some odd looking triangles though.



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Acknowledgments

- *An Introduction to Non-Euclidean Geometry* by David Gans
- Class notes from Non-Euclidean Geometry at Bob Jones University, Spring 2008 with instructor Larry Lemon
- Graphics created by Matt Black using Blender